spl Documentation

Release 1

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Contents:

SPL is a Python/Fortran 2003 library for B-Splines/NURBS and Computer Aided Design Algorithms.

SPL can be used in three different ways:

1. *Fortran 90/95* subroutines through the file **fortran/src/bsplines/bspline.F90**
2. *Fortran 2003* objects, mainly the **mapping** and **cad** objects
3. The same objects as in *2.* but through *Python*

CHAPTER 1

First Steps with SPL

This document is meant to give a tutorial-like overview of SPL.

The green arrows designate “more info” links leading to advanced sections about the described task.

By reading this tutorial, you’ll be able to:

- compile a simple *SPL* file
- get familiar with parallel programing paradigms
- create, modify and build a *SPL* project.

1.1 Install SPL

1.2 Examples

In this section, we describe some features of *SPL* on simple examples.

See `script`.

CHAPTER 2

Dive into SPL

2.1 Contents

2.1.1 Introduction

2.1.2 Input and Output

CHAPTER 3

B-Splines and NURBS

We start this section by recalling some basic properies about B-splines curves and surfaces. We also recall some fundamental algorithms (knot insertion and degree elevation).

For a basic introduction to the subject, we refer to the books [LP95] and [Far02].

A B-Splines family, $(N_i)_{1 \leqslant i \leqslant n}$ of order k, can be generated using a non-decreasing sequence **of knots** $T = (t_i)_{1 \leqslant i \leqslant n+k}$.

3.1 B-Splines series

The j-th B-Spline of order k is defined by the recurrence relation:

$$N_j^k = w_j^k N_j^{k-1} + (1 - w_{j+1}^k) N_{j+1}^{k-1}$$

where,

$$w_j^k(x) = \frac{x - t_j}{t_{j+k-1} - t_j} \qquad N_j^1(x) = \chi_{[t_j, t_{j+1}[}(x)$$

for $k \geq 1$ and $1 \leq j \leq n$.

We note some important properties of a B-splines basis:

- B-splines are piecewise polynomial of degree $p = k - 1$,
- Compact support; the support of N_j^k is contained in $[t_j, t_{j+k}]$,
- If $x \in\,]t_j, t_{j+1}[$, then only the *B-splines* $\{N_{j-k+1}^k, \cdots, N_j^k\}$ are non vanishing at x,
- Positivity: $\forall j \in \{1, \cdots, n\} \;\; N_j(x) > 0, \;\; \forall x \in]t_j, t_{j+k}[$,
- Partition of unity $\sum_{i=1}^n N_i^k(x) = 1, \forall x \in \mathbb{R}$,
- Local linear independence,
- If a knot t_i has a multiplicity m_i then the B-spline is $\mathcal{C}^{(p-m_i)}$ at t_i.

3.2 Knots vector families

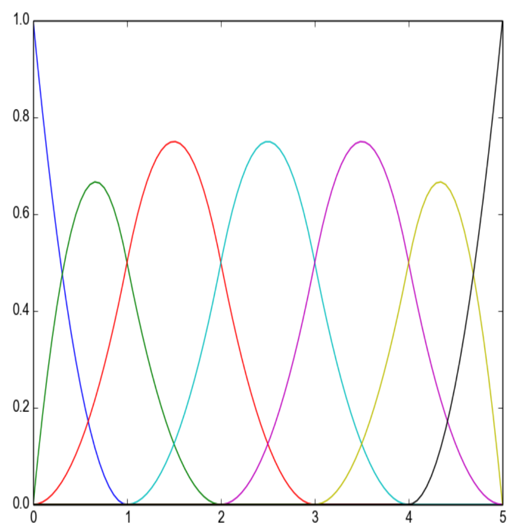
There are two kind of **knots vectors**, called **clamped** and **unclamped**. Both families contains **uniform** and **non-uniform** sequences.

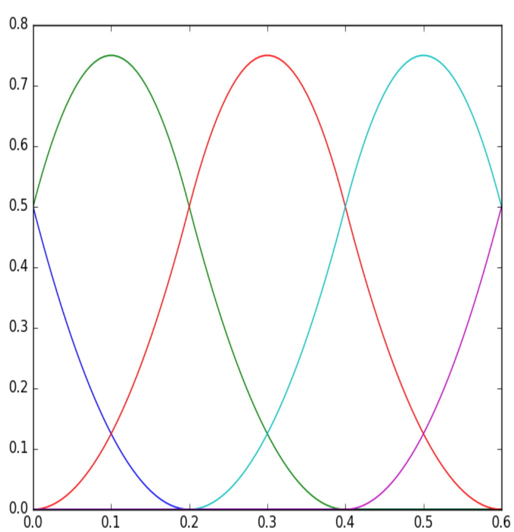
The following are examples of such knots vectors

1. **Clamped knots** (open knots vector)

- uniform

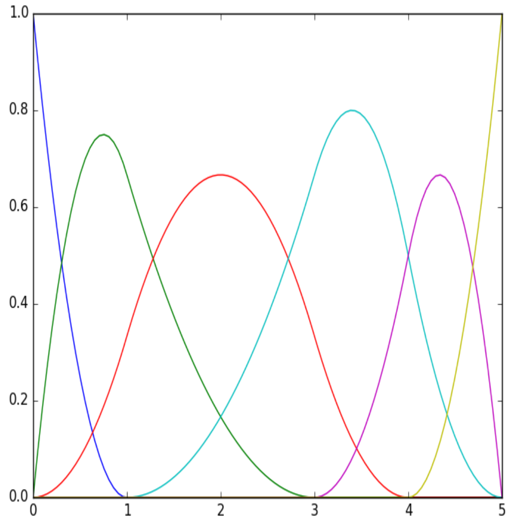
$$T_1 = \{0, 0, 0, 1, 2, 3, 4, 5, 5, 5\}$$
$$T_2 = \{-0.2, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8, 0.8\}$$

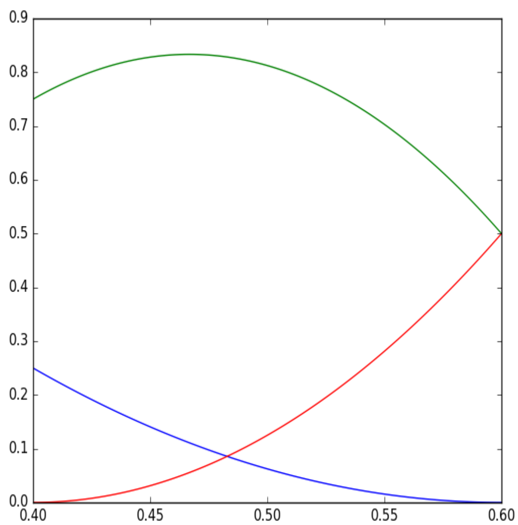




- non-uniform

$$T_3 = \{0, 0, 0, 1, 3, 4, 5, 5, 5\}$$
$$T_4 = \{-0.2, -0.2, 0.4, 0.6, 0.8, 0.8\}$$

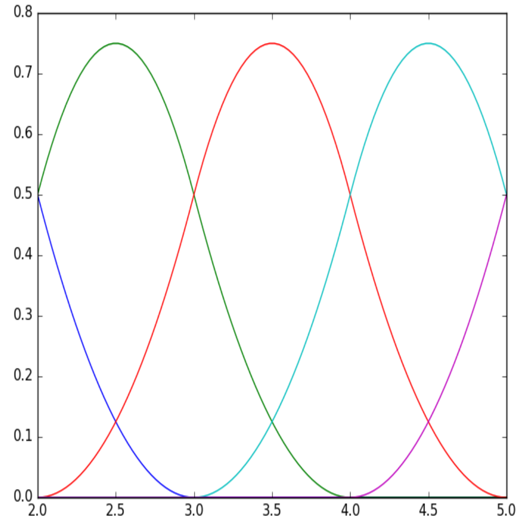


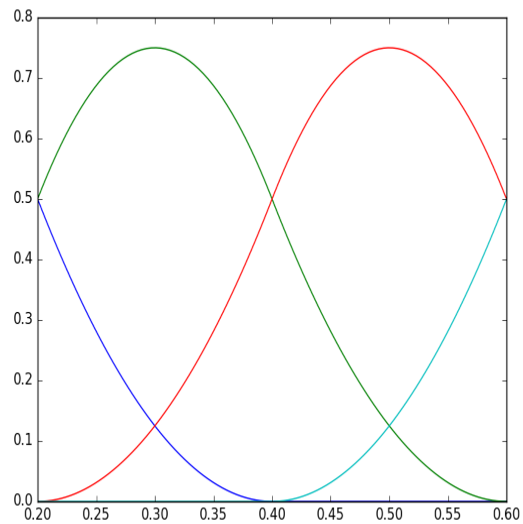


2. **Unclamped knots**

- uniform

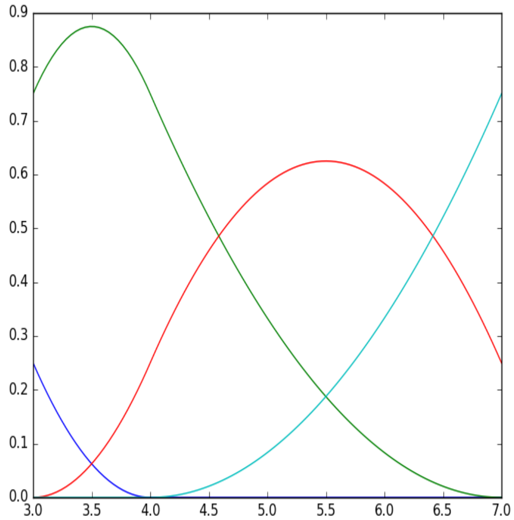
$$T_5 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$
$$T_6 = \{-0.2, 0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$$

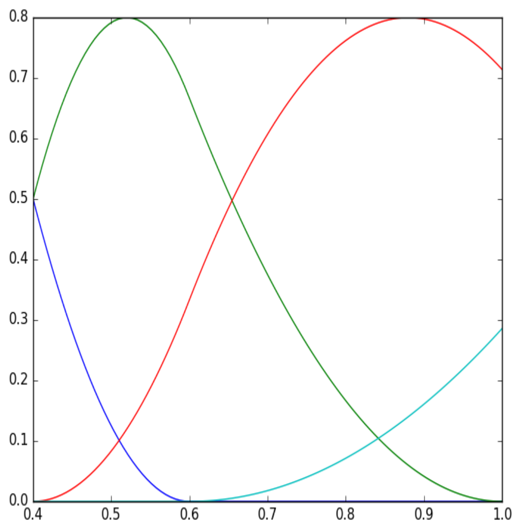




- non-uniform

$$T_7 = \{0, 0, 3, 4, 7, 8, 9\}$$
$$T_8 = \{-0.2, 0.2, 0.4, 0.6, 1.0, 2.0, 2.5\}$$



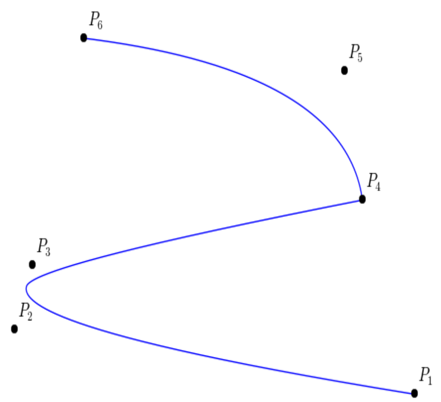


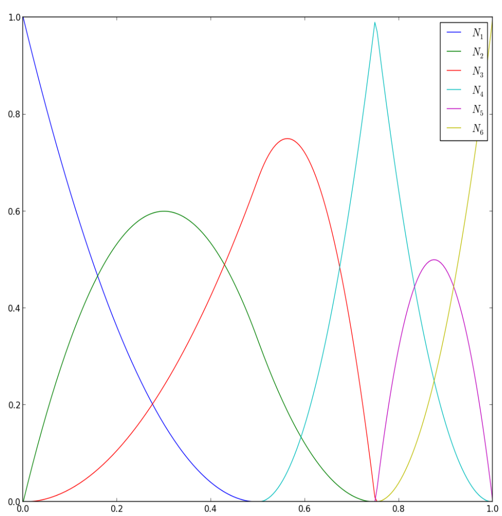
3.3 B-Spline curve

The B-spline curve in $\mathbb{R}^d$ associated to knots vector $T = (t_i)_{1 \leqslant i \leqslant n+k}$ and the control polygon $(\mathbf{P}_i)_{1 \leqslant i \leqslant n}$ is defined by :

$$\mathcal{C}(t) = \sum_{i=1}^{n} N_i^k(t)\mathbf{P}_i$$

In (Fig. ref{figBSplineCurve}), we give an example of a quadratic B-Spline curve, and its corresponding knot vector and control points.





We have the following properties for a *B-spline* curve:

- If $n = k$, then $\mathcal{C}$ is just a B'ezier-curve,
- $\mathcal{C}$ is a piecewise polynomial curve,
- The curve interpolates its extremas if the associated multiplicity of the first and the last knot are maximum (*i.e.* equal to k), *i.e.* open knot vector,
- Invariance with respect to affine transformations,
- Strong convex-hull property:

if $t_i \leq t \leq t_{i+1}$, then $\mathcal{C}(t)$ is inside the convex-hull associated to the control points $\mathbf{P}_{i-p}, \cdots, \mathbf{P}_i$,

- Local modification : moving the i^{th} control point $\mathbf{P}_i$ affects $\mathcal{C}(t)$, only in the interval $[t_i, t_{i+k}]$,
- The control polygon approaches the behavior of the curve.

Note: In order to model a singular curve, we can use multiple control points : $\mathbf{P}_i = \mathbf{P}_{i+1}$.

3.4 Multivariate tensor product splines

Let us consider d knot vectors $\mathcal{T} = \{T^1, T^2, \cdots, T^d\}$. For simplicity, we consider that these knot vectors are open, which means that k knots on each side are duplicated so that the spline is interpolating on the boundary, and of bounds 0 and 1. In the sequel we will use the notation $I = [0, 1]$. Each knot vector T^i, will generate a basis for a Schoenberg space, $\mathcal{S}_{k_i}(T^i, I)$. The tensor product of all these spaces is also a Schoenberg space, namely $\mathcal{S}_{\mathbf{k}}(\mathcal{T})$, where $\mathbf{k} = \{k_1, \cdots, k_d\}$. The cube $\mathcal{P} = I^d = [0, 1]^d$, will be referred to as a patch.

The basis for $\mathcal{S}_{\mathbf{k}}(\mathcal{T})$ is defined by a tensor product :

$$N_{\mathbf{i}}^{\mathbf{k}} := N_{i_1}^{k_1} \otimes N_{i_2}^{k_2} \otimes \cdots \otimes N_{i_d}^{k_d}$$

where, $\mathbf{i} = \{i_1, \cdots, i_d\}$.

A typical cell from $\mathcal{P}$ is a cube of the form : $Q_{\mathbf{i}} = [\xi_{i_1}, \xi_{i_1+1}] \otimes \cdots \otimes [\xi_{i_d}, \xi_{i_d+1}]$.

3.5 Deriving a B-spline curve

The derivative of a B-spline curve is obtained as:

$$\mathcal{C}'(t) = \sum_{i=1}^{n} N_i^{k'}(t)\mathbf{P}_i = \sum_{i=1}^{n} \left(\frac{p}{t_{i+p} - t_i} N_i^{k-1}(t)\mathbf{P}_i - \frac{p}{t_{i+1+p} - t_{i+1}} N_{i+1}^{k-1}(t)\mathbf{P}_i \right) = \sum_{i=1}^{n-1} N_i^{k-1^*}(t)\mathbf{Q}_i$$

where $\mathbf{Q}_i = p\frac{\mathbf{P}_{i+1} - \mathbf{P}_i}{t_{i+1+p} - t_{i+1}}$, and $\{N_i^{k-1^*},\ \ 1 \leq i \leq n-1\}$ are generated using the knot vector T^*, which is obtained from T by reducing by one the multiplicity of the first and the last knot (in the case of open knot vector), *i.e.* by removing the first and the last knot.

More generally, by introducing the B-splines family $\{N_i^{k-j^*}, 1 \leq i \leq n-j\}$ generated by the knots vector T^{j^*} obtained from T by removing the first and the last knot j times, we have the following result:

3.5.1 proposition

The j^{th} derivative of the curve $\mathcal{C}$ is given by

$$\mathcal{C}^{(j)}(t) = \sum_{i=1}^{n-j} N_i^{k-j^*}(t)\mathbf{P}_i^{(j)},$$

where, for $j > 0$

$$\mathbf{P}_i^{(j)} = \frac{p-j+1}{t_{i+p+1} - t_{i+j}} \left(\mathbf{P}_{i+1}^{(j-1)} - \mathbf{P}_i^{(j-1)} \right)$$

$$\text{and} \quad \mathbf{P}_i^{(0)} = \mathbf{P}_i.$$

By denoting $\mathcal{C}'$ and $\mathcal{C}''$ the first and second derivative of the B-spline curve $\mathcal{C}$, it is easy to show that:

We have,

- $\mathcal{C}'(0) = \frac{p}{t_{p+2}} (\mathbf{P}_2 - \mathbf{P}_1)$,

- $\mathcal{C}'(1) = \frac{p}{1-t_n}\left(\mathbf{P}_n - \mathbf{P}_{n-1}\right)$,
- $\mathcal{C}''(0) = \frac{p(p-1)}{t_{p+2}}\left(\frac{1}{t_{p+2}}\mathbf{P}_1 - \{\frac{1}{t_{p+2}} + \frac{1}{t_{p+3}}\}\mathbf{P}_2 + \frac{1}{t_{p+3}}\mathbf{P}_3\right)$,
- $\mathcal{C}''(1) = \frac{p(p-1)}{1-t_n}\left(\frac{1}{1-t_n}\mathbf{P}_n - \{\frac{1}{1-t_n} + \frac{1}{1-t_{n-1}}\}\mathbf{P}_{n-1} + \frac{1}{1-t_{n-1}}\mathbf{P}_{n-2}\right)$.

3.5.2 Example

Let us consider the quadratic B-spline curve associated to the knots vector $T = \{000\ \frac{2}{5}\ \frac{3}{5}\ 111\}$ and the control points $\{P_i, 1 \leq i \leq 5\}$:

$$\mathcal{C}(t) = \sum_{i=1}^{5} N_i^{3'}(t)\mathbf{P}_i$$

we have,

$$\mathcal{C}'(t) = \sum_{i=1}^{4} N_i^{2^*}(t)\mathbf{Q}_i$$

where

$$\mathbf{Q}_1 = 5\{\mathbf{P}_2 - \mathbf{P}_1\}, \quad \mathbf{Q}_2 = \frac{10}{3}\{\mathbf{P}_3 - \mathbf{P}_2\},$$
$$\mathbf{Q}_3 = \frac{10}{3}\{\mathbf{P}_4 - \mathbf{P}_3\}, \quad \mathbf{Q}_4 = 5\{\mathbf{P}_5 - \mathbf{P}_4\}.$$

The *B-splines* $\{N_i^{2^*},\ 1 \leq i \leq 4\}$ are associated to the knot vector $T^* = \{00\ \frac{2}{5}\ \frac{3}{5}\ 11\}$.

Fundamental geometric operations

By inserting new knots into the knot vector, we add new control points without changing the shape of the B-Spline curve. This can be done using the DeBoor algorithm [dB01]. We can also elevate the degree of the B-Spline family and keep unchanged the curve [HHM05]. In (Fig. ref{refinement_curve_B_Spline}), we apply these algorithms on a quadratic B-Spline curve and we show the position of the new control points.

3.6 Knot insertion

After modification, we denote by $\widetilde{n}, \widetilde{k}, \widetilde{T}$ the new parameters. $(\mathbf{Q}_i)$ are the new control points.

One can insert a new knot t, where $t_j \leqslant t < t_{j+1}$. For this purpose we use the DeBoor algorithm [dB01]:

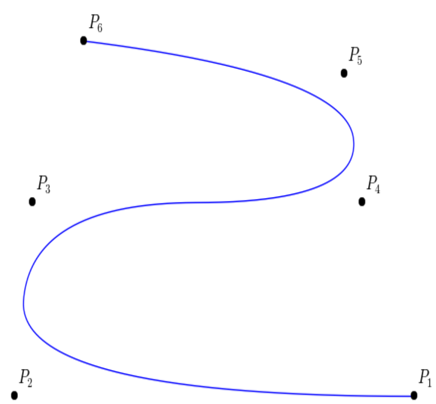
$$\begin{aligned}
\widetilde{n} &= n+1 \\
\widetilde{k} &= k \\
\widetilde{T} &= \{t_1, .., t_j, t, t_{j+1}, .., t_{n+k}\} \\
\alpha_i = \left\{ \begin{array}{cc} 1 & 1 \leqslant i \leqslant j-k+1 \\ \frac{t-t_i}{t_{i+k-1}-t_i} & j-k+2 \leqslant i \leqslant j \\ 0 & j+1 \leqslant i \end{array} \right. \\
\mathbf{Q}_i &= \alpha_i \mathbf{P}_i + (1-\alpha_i)\mathbf{P}_{i-1}
\end{aligned}$$

Many other algorithms exist, like blossoming for fast insertion algorithm. For more details about this topic, we refer to [NT93].

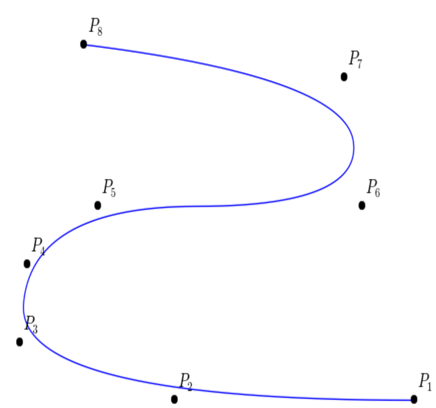
3.7 Order elevation

We can elevate the order of the basis, without changing the curve. Several algorithms exist for this purpose. We used the one by Huang et al. [PP91], [HHM05].

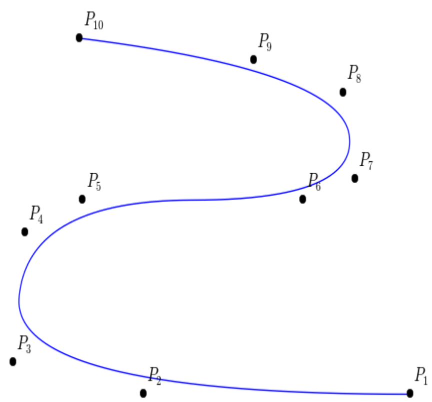
A quadratic B-spline curve and its control points. The knot vector is $T = \{000, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 111\}$.



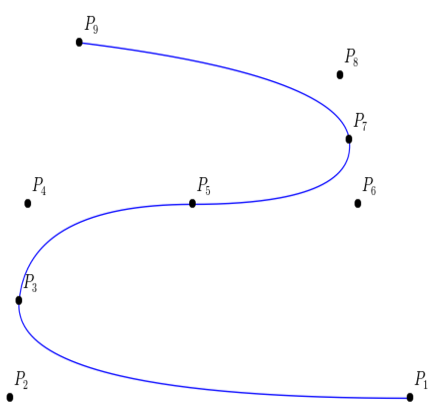
The curve after a h-refinement by inserting the knots $\{0.15, 0.35\}$ while the degree is kept equal to 2.



The curve after a p-refinement, the degree was raised by 1 (using cubic B-splines).



The curve after duplicating the multiplicity of the internal knots $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, this leads to a B'ezier description. We can then, split the curve into 4 pieces (sub-domains), each one will corresponds to a quadratic B'ezier curve.



3.8 Translation

3.9 Rotation

Todo: not yet available

3.10 Scaling

Todo: not yet available

References

CHAPTER 4

GLT

4.1 Where do the GLTs come from?

The main aim of this paragraph is to present a crucial example that highlights the importance of the GLT algebra when dealing with linear systems coming from the discretization of PDEs. Let us start with some preliminaries. In detail, we will recall the notion of symbol of a matrix-sequence and the basic idea behind the GLT theory.

4.1.1 Spectral preliminaries

The following one is a rather informal definition of symbol of a matrix-sequence.

example:

When $d_n = n$, $d = 1$, $D = [0, \pi]$, $\{A_n\}_n \sim_\lambda f$ means

References

CHAPTER 5

Exterior Algebra

Let V be a real vector space of dimension n.

Definition, Alternating algebraic forms:

For each k, we define $\mathrm{Alt}^k V$ as the space of alternating k-linear maps $V \times \cdots \times V \to \mathbb{R}$.

Note:

- $\mathrm{Alt}^0 = \mathbb{R}$,
- $\mathrm{Alt}^1 = V^*$ is the dual space of V (the space of covectors)

Definition, Exterior product:

For $\omega \in \mathrm{Alt}^j$ and $\eta \in \mathrm{Alt}^k$, their exterior (wedge) product is given by:

$$(\omega \wedge \eta)(v_1, \cdots, v_{j+k}) = \sum_{\sigma} (\text{sign } \sigma) \omega(v_{\sigma(1)}, \cdots, v_{\sigma(j)}) \eta(v_{\sigma(j+1)}, \cdots, v_{\sigma(j+k)})$$

for all $v_i \in V$. Where the sum is over all permutations σ of $\{1, \cdots, j+k\}$, for which $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(j+k)$.

Note:

- The exterior product is **bilinear**, **associative**,
- **anti-commutative**: $\eta \wedge \omega = (-1)^{jk} \omega \wedge \eta$ for all $\omega \in \mathrm{Alt}^j$ and $\eta \in \mathrm{Alt}^k$.

Definition, Grassmann Algebra:

Grassmann Algebra is defined by:

$$\mathrm{Alt}\, V := \bigoplus_k \mathrm{Alt}^{\,k} V$$

This is a **anti-commutative graded algebra**. Also called **Exterior Algebra** of V^*

In the case of $V = \mathbb{R}^n$, we have:

- $\mathrm{Alt}\, V^0 \sim \mathbb{R}$,
- $\mathrm{Alt}\, V^1 \sim \mathbb{R}^n$,
- $\mathrm{Alt}\, V^{n-1} \sim \mathbb{R}^n$, using Riesz representation theorem,
- $\mathrm{Alt}\, V^n \sim \mathbb{R}$, using the map $v \longmapsto \det(v, v_1, \cdots, v_{n-1})$.

5.1 Basis

Let $v_1, \cdots, v_n$ be a basis of V and $\mu_1, \cdots, \mu_n$ the associated dual basis for V^* ($\mu_i(v_j) = \delta_{ij}$).

For any increasing permutations $\sigma, \rho : \{1, \cdots, k\} \longrightarrow \{1, \cdots, n\}$, we have:

$$\mu_{\sigma(1)} \wedge \cdots \wedge \mu_{\sigma(k)}(v_{\rho(1)}, \cdots, v_{\rho(k)}) = \chi_{\sigma,\rho}$$

thus the $\binom{n}{k}$ algebraic k-forms $\mu_{\sigma(1)} \wedge \cdots \wedge \mu_{\sigma(k)}$, form a basis for $\mathrm{Alt}^{\,k} V$ and $\dim \mathrm{Alt}^{\,k} V = \binom{n}{k}$.

Definition, Interior product:

Let ω be a k-form, and $v \in V$. The **interior product** of ω and v is the $(k-1)$-form $\omega \lrcorner v$ defined by:

$$\omega \lrcorner v(v_1, \cdots, v_{k-1}) = \omega(v, v_1, \cdots, v_{k-1})$$

- We have for $\omega \in \mathrm{Alt}^{\,k} V$, $\eta \in \mathrm{Alt}^{\,l} V$ and $v \in V$:

$$(\omega \wedge \eta) \lrcorner v = (\omega \lrcorner v) \wedge \eta + (-1)^k \omega \wedge (\eta \lrcorner v)$$

Definition, Inner product:

If V is has an inner product, then $\mathrm{Alt}^{\,k} V$ is endowed with an inner product given by:

$$(\omega, \eta) = \sum_\rho \omega(v_{\rho(1)}, \cdots, v_{\rho(k)}) \eta(v_{\rho(1)}, \cdots, v_{\rho(k)}), \quad \forall \omega, \eta \in \mathrm{Alt}^{\,k} V.$$

where the sum is over increasing sequences $\rho : \{1, \cdots, k\} \longrightarrow \{1, \cdots, n\}$, and $v_1, \cdots, v_n$ is any orthonormal basis.

5.2 Orientation and Volume form

Todo: add Orientation and Volume form

Definition, Pullback:

A linear transformation of vector spaces $L : V \to W$ induces a transformation $L^* : \mathrm{Alt}\, W \to \mathrm{Alt}\, V$, called the **pullback**, and given by:

$$L^*\omega(v_1, \cdots, v_k) = \omega(Lv_1, \cdots, Lv_k), \quad \forall\omega \in \mathrm{Alt}^{\,k} W, \quad v_1, \cdots, v_k \in V$$

- The pullback acts **contravariantly**: if $U \xrightarrow{K} V \xrightarrow{L} W$ then,

$$\mathrm{Alt}\, W \xrightarrow{K^*} \mathrm{Alt}\, V \xrightarrow{L^*} \mathrm{Alt}\, U$$

- $L^*(\omega \wedge \eta) = L^*\omega \wedge L^*\eta$

Let V be a subspace of W. For the inclusion $\imath_V : V \longrightarrow W$, we can define its pullback $\imath_V^*$: this is a **surjection** of $\mathrm{Alt}\, W$ onto $\mathrm{Alt}\, V$.

If W has an inner product and $\pi_V : W \longrightarrow V$ is the orthogonal projection. We can define its pullback π_V^* : this an **injection** of $\mathrm{Alt}\, V$ onto $\mathrm{Alt}\, W$.

Let us consider the composition : W shortstack{π_V \ $\longrightarrow$} V shortstack{$\imath_V$ \ $\longrightarrow$} W, and its pullback $\pi_V^* \imath_V^*$.

Definition, The tangential and normal parts:

- $\pi_V^* \imath_V^*$ associates for each $\omega \in \mathrm{Alt}^{\,k}$ its **tangential** part $\omega_\parallel$ with respect to V :

$$(\pi_V^* \imath_V^* \omega)(v_1, \cdots, v_k) = \omega(\pi_V v_1, \cdots, \pi_V v_k), \quad \forall v_1, \cdots, v_k \in W.$$

- $\omega - \pi_V^* \imath_V^* \omega$ associates for each $\omega \in \mathrm{Alt}^{\,k}$ its **normal** part $\omega_\perp$ with respect to V.

The **tangential part** of ω vanishes if and only if the image of ω in $\mathrm{Alt}^{\,k} V$ vanishes.

Let V be an oriented inner product space, with volume form vol. Let $\omega \in \mathrm{Alt}^{\,k} V$. We can define a new linear map L_ω as the composition of $\mathrm{Alt}^{\,n-k} V \longrightarrow \mathrm{Alt}^{\,n} V$ such as:

$$\mu \longmapsto \omega \wedge \mu$$

and the canonical isomorphism of $\mathrm{Alt}^{\,n} V$ onto $\mathbb{R}$, and using the Riesz representation theorem, there exists an element $\star\omega \in \mathrm{Alt}^{\,n-k} V$ such that : $L_\omega(\mu) = (\star\omega, \mu)$, *i.e.*:

$$\omega \wedge \mu = (\star\omega, \mu)\mathrm{vol}, \quad \omega \in \mathrm{Alt}^{\,k}, \; \mu \in \mathrm{Alt}^{\,n-k}$$

Definition, The Hodge star operation:

The linear map which maps $\mathrm{Alt}^{\,k} V$ onto $\mathrm{Alt}^{\,n-k} V$ $\omega \longmapsto \star\omega$ is called the **Hodge star** operator.

- If $e_1, \cdots, e_n$ is any positively oriented orthonormal basis, and σ a permutation, we have

$$\omega(e_{\sigma(1)}, \cdots, e_{\sigma(k)}) = (\mathrm{sign}\sigma) \star \omega(e_{\sigma(k+1)}, \cdots, e_{\sigma(n)})$$

- $\star \star \omega = (-1)^{k(n-k)}\omega, \quad \forall\omega \in \mathrm{Alt}^{\,k} V$, thus the Hodge star is an **isometry**.

- $(\star\omega)_{\|} = \star(\omega_{\perp})$ and $(\star\omega)_{\perp} = \star(\omega_{\|})$
- the image of $\star\omega$ in $\mathrm{Alt}^{k}V$ vanishes if and only if $\omega_{\perp}$ vanishes.

$\mathrm{Alt}^0\mathbb{R}^3 \cong \mathbb{R}$ $\mathrm{Alt}^1\mathbb{R}^3 \cong \mathbb{R}^3$ $\mathrm{Alt}^2\mathbb{R}^3 \cong \mathbb{R}^3$ $\mathrm{Alt}^3\mathbb{R}^3 \cong \mathbb{R}$	$c \leftrightarrow c$ $u_1\mathrm{d}\,x_1 + u_2\mathrm{d}\,x_2 + u_3\mathrm{d}\,x_3 \leftrightarrow u$ $u_3\mathrm{d}\,x_1 \wedge \mathrm{d}\,x_2 - u_2\mathrm{d}\,x_1 \wedge \mathrm{d}\,x_3 + u_1\mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3 + \leftrightarrow u$ $c\mathrm{d}\,x_1 \wedge \mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3 \leftrightarrow c$

$\wedge : \mathrm{Alt}^1\mathbb{R}^3 \times \mathrm{Alt}^1\mathbb{R}^3 \longrightarrow \mathrm{Alt}^2\mathbb{R}^3$ $\wedge : \mathrm{Alt}^1\mathbb{R}^3 \times \mathrm{Alt}^2\mathbb{R}^3 \longrightarrow \mathrm{Alt}^3\mathbb{R}^3$	$\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$L^* : \mathrm{Alt}^0\mathbb{R}^3 \longrightarrow \mathrm{Alt}^0\mathbb{R}^3$ $L^* : \mathrm{Alt}^1\mathbb{R}^3 \longrightarrow \mathrm{Alt}^1\mathbb{R}^3$ $L^* : \mathrm{Alt}^2\mathbb{R}^3 \longrightarrow \mathrm{Alt}^2\mathbb{R}^3$ $L^* : \mathrm{Alt}^3\mathbb{R}^3 \longrightarrow \mathrm{Alt}^3\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ $L^T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $(\det L)L^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $(\det L) : \mathbb{R} \longrightarrow \mathbb{R} \quad (c \longmapsto c \det L)$

$\lrcorner v : \mathrm{Alt}^1\mathbb{R}^3 \longrightarrow \mathrm{Alt}^0\mathbb{R}^3$ $\lrcorner v : \mathrm{Alt}^2\mathbb{R}^3 \longrightarrow \mathrm{Alt}^1\mathbb{R}^3$ $\lrcorner v : \mathrm{Alt}^3\mathbb{R}^3 \longrightarrow \mathrm{Alt}^2\mathbb{R}^3$	$v\cdot : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $v\times : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $v : \mathbb{R} \longrightarrow \mathbb{R}^3 \quad (c \longmapsto cv)$

inner product on $\mathrm{Alt}^k\mathbb{R}^3$ induced by dot product on $\mathbb{R}^3$ $\mathrm{vol} = \mathrm{d}\,x_1 \wedge \mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3$	dot product on $\mathbb{R}$ and $\mathbb{R}^3$ $(v_1, v_2, v_3) \longmapsto \det(v_1\vert v_2\vert v_3)$

$\star : \mathrm{Alt}^0\mathbb{R}^3 \longrightarrow \mathrm{Alt}^3\mathbb{R}^3$ $\star : \mathrm{Alt}^1\mathbb{R}^3 \longrightarrow \mathrm{Alt}^2\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}$ $\mathrm{id} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

5.3 Exterior Calculus on manifolds and Differential forms

Let Ω be a smooth manifold, of dimension n.

- $\forall x \in \Omega$ we denote by $T_x\Omega$ the tangent space. This is a vector space of dimension n,
- tangent bundle $\{(x, v), \ \ x \in \Omega, v \in T_x\Omega\}$,
- Applying the exterior algebra to the tangent spaces, we obtain the exterior forms bundle, whose elements are pairs (x, μ) with $x \in \Omega$ and $\mu \in \mathrm{Alt}^{k}T_x\Omega$.
- a **differential** k-form ω is a section of this bundle. This is a map which associates to each $x \in \Omega$ an element $\omega_x \in \mathrm{Alt}^{k}T_x\Omega$,
- if the map $\mathcal{L}^k_\omega : x \longmapsto \omega_x(v_1(x), \cdots, v_k(x))$ is smooth (whenever v_i are smooth), we say that ω is a smooth differential k-form,
- we define $\Lambda^k(\Omega)$ the space of all smooth k-forms on Ω,
- $\Lambda^0(\Omega) = \mathcal{C}^\infty(\Omega)$,
- if the map $\mathcal{L}^k_\omega$ is $\mathcal{C}^m(\Omega)$, we define differential k-forms with less smoothness $\mathcal{C}^m\Lambda^k(\Omega)$.

Let Ω be a smooth manifold, of dimension n.

Exterior product:

if $\omega \in \Lambda^k(\Omega)$ and $\eta \in \Lambda^j(\Omega)$, we may define $\omega \wedge \eta$ as $(\omega \wedge \eta)_x = \omega_x \wedge \eta_x$ and the Grassmann algebra $\Lambda(\Omega) := \bigoplus_k \Lambda^k(\Omega)$

Differential forms can be differentiated and integrated, without recourse to any additional structure, such as a metric or a measure.

Exterior differentiation:

For each $\omega \in \Lambda^k(\Omega)$, can define the $(k+1)$-form $\mathrm{d}\,\omega \in \Lambda^{k+1}(\Omega)$, such as:

$$\mathrm{d}\,\omega_x(v_1, \cdots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^j \partial_{v_j} \omega_x(v_1, \cdots, \hat{v_j}, \cdots, v_{k+1})$$

where the hat is used to indicated a suppressed argument.

This defines a graded linear operator of degree $+1$, of $\Lambda(\Omega)$ onto $\Lambda(\Omega)$.

We have the following properties:

- $\mathrm{d} \circ \mathrm{d} = 0$
- $\mathrm{d}\,(\omega \wedge \eta) = \mathrm{d}\,\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\,\eta, \quad \forall \omega \in \Lambda^k(\Omega), \eta \in \Lambda^j(\Omega)$,
- (Pullback) let ϕ be a smooth map of Ω onto Ω'. Then $\phi^*(\omega \wedge \eta) = \phi^*(\omega) \wedge \phi^*(\eta)$ and $\phi^*(\mathrm{d}\,\omega) = \mathrm{d}\,(\phi^* \omega)$,
- (Interior product) the interior product of a differential k-form ω with a vector field v,
- we obtain a $(k-1)$-form by : $(\omega \lrcorner v)_x := \omega_x \lrcorner v_x$,
- (Trace operator) the pullback $i^*_{\partial\Omega}$ of $i_{\partial\Omega}$ is the trace operator Tr

Integration:

- If f is an oriented, piecewise smooth k-dimensional submanifold of Ω, and ω is a continuous k-form, then th integral $\int_f \omega$ is well defined :
  - [0-forms] can be evaluated at points,
  - [1-forms] can be integrated over directed curves,
  - [2-forms] can be integrated over directed surfaces,
- (Inner product) The L^2-inner product of two differential k-forms on an oriented Riemannian manifold Ω is defined as :

$$(\omega, \eta)_{L^2\Lambda^k} = \int_\Omega (\omega_x, \eta_x) \mathrm{vol} = \int \omega \wedge \star\eta$$

The completion of $\Lambda^k(\Omega)$ in the corresponding norm defines the Hilbert space $L^2\Lambda^k(\Omega)$.

We have the following results:

- (Integration) if ϕ is an orientation-preserving diffeomorphism, then

$$\int_\Omega \phi^*\omega = \int_{\Omega'} \omega, \quad \forall \omega \in \Lambda^n(\Omega')$$

Theorem, Stokes theorem:

If Ω is an oriented n-manifold with boundary $\partial\Omega$, then

$$\int_\Omega \mathrm{d}\,\omega = \int_{\partial\Omega} \mathrm{Tr}\,\omega, \quad \forall \omega \in \Lambda^{n-1}(\Omega)$$

Theorem, Integration by parts:

If Ω is an oriented n-manifold with boundary $\partial\Omega$, then

$$\int_\Omega \mathrm{d}\,\omega \wedge \eta = (-1)^{k-1} \int_\Omega \omega \wedge \mathrm{d}\,\eta + \int_{\partial\Omega} \mathrm{Tr}\,\omega \wedge \mathrm{Tr}\,\eta, \quad \forall \omega \in \Lambda^k(\Omega), \eta \in \Lambda^{n-k-1}(\Omega)$$

5.4 Sobolev spaces of differential forms

As for the classical case, we can define the Sobolev spaces as:

- $H^s\Lambda^k(\Omega)$ is the space of differential k-forms such that $\mathcal{L}^k_\omega \in H^s(\Omega)$.
- $H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega),\ \mathrm{d}\,\omega \in L^2\Lambda^{k+1}(\Omega)\}$. The associated norm is :

$$\|\omega\|^2_{H\Lambda^k} = \|\omega\|^2_{H\Lambda} := \|\omega\|^2_{L^2\Lambda^k} + \|\mathrm{d}\,\omega\|^2_{L^2\Lambda^{k+1}}$$

- $H\Lambda^0(\Omega)$ coincides with $H^1\Lambda^0(\Omega)$,
- $H\Lambda^n(\Omega)$ coincides with $L^2\Lambda^n(\Omega)$,
- for $0 < k < n$, we have $H^1\Lambda^k(\Omega) \subset H\Lambda^k(\Omega) \subset L^2\Lambda^k(\Omega)$, strictly.

k	Λ^k	$H\Lambda^k$	$\mathrm{d}\,\omega$	$\int_f \omega$	$\kappa\omega$
0	$\mathcal{C}^\infty$	H^1	$\nabla\omega$	$\omega(f)$	0
1	$\mathcal{C}^\infty(\mathbb{R}^3)$	$H(\mathrm{rot}\,,\mathbb{R}^3)$	$\mathrm{rot}\,\omega$	$\int_f \omega \cdot t\mathrm{d}\,\mathcal{H}_1$	$x \longmapsto x \cdot \omega(x)$
2	$\mathcal{C}^\infty(\mathbb{R}^3)$	$H(\mathrm{div}\,,\mathbb{R}^3)$	$\mathrm{div}\,\omega$	$\int_f \omega \cdot n\mathrm{d}\,\mathcal{H}_2$	$x \longmapsto x \times \omega(x)$
3	$\mathcal{C}^\infty$	L^2	0	$\int_f \omega\mathrm{d}\,\mathcal{H}_3$	$x \longmapsto x\omega(x)$

5.5 Cohomology and De Rham Complex

The De Rham complex is the sequence of spaces and mappings

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{\mathrm{d}} \Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^n(\Omega) \longrightarrow 0$$

Since, $\mathrm{d}\,\circ \mathrm{d}\, = 0$, we have

$$\mathcal{R}(\mathrm{d}\,: \Lambda^{k-1}(\Omega) \longrightarrow \Lambda^k(\Omega)) \subset \mathcal{N}(\mathrm{d}\,: \Lambda^k(\Omega) \longrightarrow \Lambda^{k+1}(\Omega))$$

If Ω is an oriented Riemannian manifold, we have the following cohomology:

$$0 \longrightarrow H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}} H\Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H\Lambda^n(\Omega) \longrightarrow 0$$

The *coderivative operator* $\delta : \Lambda^k(\Omega) \longrightarrow \Lambda^{k-1}(\Omega)$ is defined as:

$$\star\delta\omega = (-1)^k \mathrm{d}\,\star\omega, \quad \omega \in \Lambda^k(\Omega)$$

- we have

$$(\mathrm{d}\,\omega, \eta) = (\omega, \delta\eta) + \int_{\partial\Omega} \mathrm{Tr}\,\omega \wedge \mathrm{Tr}\,\eta, \quad \forall\omega \in \Lambda^k(\Omega), \eta \in \Lambda^{k+1}(\Omega),$$

- δ is a graded linear operator of degree -1.
- δ is the formal adjoint of d whenever ω or η vanishes near the boundary.
- we define the spaces

$$H^*\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega), \ \ \delta\omega \in L^2\Lambda^{k-1}(\Omega)\}.$$

  we have $H^*\Lambda^k(\Omega) = \star H\Lambda^{n-k}(\Omega)$.

- we obtain the dual complex

$$0 \longleftarrow H^*\Lambda^0(\Omega) \xleftarrow{\delta} H^*\Lambda^1(\Omega) \xleftarrow{\delta} \cdots \xleftarrow{\delta} H^*\Lambda^n(\Omega) \longleftarrow 0$$

5.6 Cohomology with boundary conditions

Let $\Lambda_0^k(\Omega)$ be the subspace of $\Lambda^k(\Omega)$ of smooth k-forms with compact support. We have $\mathrm{d}\,\Lambda_0^k \subset \Lambda_0^{k+1}$.

The De Rham complex with the compact support is

$$0 \longrightarrow \Lambda_0^0(\Omega) \xrightarrow{\mathrm{d}} \Lambda_0^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda_0^n(\Omega) \longrightarrow 0$$

Recall that the closure of $\Lambda_0^k(\Omega)$ in $H\Lambda^k(\Omega)$ is

$$H_0\Lambda^k(\Omega) = \{\omega \in H\Lambda^k(\Omega), \ \ \mathrm{Tr}\,\omega = 0\}.$$

The L^2 version of the last complex is

$$0 \longrightarrow H_0\Lambda^0(\Omega) \xrightarrow{\mathrm{d}} H_0\Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H_0\Lambda^n(\Omega) \longrightarrow 0$$

Definition, Harmonic forms:

The harmonic k-forms are the differential k-forms that verify the differential equations

$$\begin{cases} \mathrm{d}\,\omega = & 0, \\ \delta\omega = & 0, \\ \mathrm{Tr}\,\star\omega = & 0. \end{cases}$$

this defines the following space,

$$\mathfrak{H}^k(\Omega) = \{\omega \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega), \ \ \mathrm{d}\,\omega = 0, \delta\omega = 0\}$$

We can also define the following space,

$$\mathfrak{H}_0^k(\Omega) = \{\omega \in H_0\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega), \ \ \mathrm{d}\,\omega = 0, \delta\omega = 0\}$$

As we can see, $\star\mathfrak{H}^k(\Omega) = \mathfrak{H}_0^{n-k}(\Omega)$.

Proposition, Poincaré duality:

There is an isomorphism between the k th De Rham cohomology space and the $(n-k)$ th cohomology space with boundary conditions.

5.7 Homological Algebra and Hilbert complexes

5.7.1 Homological Algebra

- A cochain complex is a sequence of vector spaces and linear maps
- k-cocycles $\mathfrak{Z}^k := \mathcal{N}(d_k)$,
- k-coboundaries $\mathfrak{B}^k := \mathcal{R}(d_{k-1})$,
- k-cohomology $\mathcal{H}^k(V) := \mathfrak{Z}^k / \mathfrak{B}^k$,
- we say that the sequence is **exact**, if the **cohomology vanishes** (*i.e.* $\forall\, k,\;\; \mathcal{H}^k(V) = \{0\}$),
- Given two cochain complexes V, V', a **cochain map** $f = (f_k)$ (such as $\mathrm{d}\,'_k f_k = f_{k+1} \mathrm{d}\,_k$)

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & V_{k-1} & \overset{\mathrm{d}\,_{k-1}}{\longrightarrow} & V_k & \overset{\mathrm{d}\,_k}{\longrightarrow} & V_{k+1} & \longrightarrow & \cdots \\
 & & \downarrow f_{k-1} & & \downarrow f_k & & \downarrow f_{k+1} & & \\
\cdots & \longrightarrow & V'_{k-1} & \overset{\mathrm{d}\,'_{k-1}}{\longrightarrow} & V'_k & \overset{\mathrm{d}\,'_k}{\longrightarrow} & V'_{k+1} & \longrightarrow & \cdots
\end{array}
$$

- f_k maps k-cochains to k-cochains and k-coboundaries to k-coboundaries, thus induces a map $\mathcal{H}^k(f) : \mathcal{H}^k(V) \longrightarrow \mathcal{H}^k(V')$.

Let $V' \subset V$ be two cochain complexes,

- The inclusion $\imath_V$ is a cochain map and thus induces a map of cohomology $\mathcal{H}^k(V') \longrightarrow \mathcal{H}^k(V)$,
- If there exists a cochain projection of V onto V', (this leads to $\pi \circ \imath = \mathrm{id}\,_{V'}$) so $\mathcal{H}^k(\pi) \circ \mathcal{H}^k(\imath) = \mathrm{id}\,_{\mathcal{H}^k(V')}$.

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & V_{k-1} & \overset{\mathrm{d}\,_{k-1}}{\longrightarrow} & V_k & \longrightarrow & \cdots \\
 & & \pi_{k-1} \downarrow\uparrow \imath & & \pi_k \downarrow\uparrow \imath & & \\
\cdots & \longrightarrow & V'_{k-1} & \overset{\mathrm{d}\,_{k-1}}{\longrightarrow} & V'_k & \longrightarrow & \cdots
\end{array}
$$

Thus, $\mathcal{H}^k(\imath)$ is **injective** and $\mathcal{H}^k(\pi)$ is **surjective**. Hence, if one of the cohomology spaces $\mathcal{H}^k(V)$ vanishes, then so does $\mathcal{H}^k(V')$

5.7.2 Cycles and boundaries of the De Rham complex

- k-cocycles

$$
\mathfrak{Z}^k = \{\omega \in H\Lambda^k(\Omega),\;\; \mathrm{d}\,\omega = 0\},\quad \mathfrak{Z}^{*k} = \{\omega \in H^*\Lambda^k(\Omega),\;\; \delta\omega = 0\},
$$

$$
\mathfrak{Z}^k_0 = \{\omega \in H_0\Lambda^k(\Omega),\;\; \mathrm{d}\,\omega = 0\},\quad \mathfrak{Z}^{*k}_0 = \{\omega \in H^*_0\Lambda^k(\Omega),\;\; \delta\omega = 0\},
$$

- k-coboundaries

$$\mathfrak{B}^k = \mathrm{d}\, H\Lambda^{k-1}(\Omega), \quad \mathfrak{B}^{*k} = \delta\Lambda^{k+1}(\Omega),$$

$$\mathfrak{B}^k_0 = \mathrm{d}\, H_0\Lambda^{k-1}(\Omega), \quad \mathfrak{B}^{*k}_0 = \delta\Lambda^{k+1}_0(\Omega),$$

- each of the spaces of cycles is closed in $\mathcal{H}\Lambda^k(\Omega)$ ($\mathcal{H}^*\Lambda^k(\Omega)$), as well in $L^2\Lambda^k(\Omega)$.
- each of the spaces of boundaries is closed in $L^2\Lambda^k(\Omega)$.
- let $\perp$ denotes the orthogonal complement in $L^2\Lambda^k(\Omega)$,

$$\mathfrak{Z}^{k\perp} \subset \mathfrak{B}^{k\perp} = \mathfrak{Z}^{*k}_0, \quad \mathfrak{Z}^{*k\perp} \subset \mathfrak{B}^{*k\perp} = \mathfrak{Z}^{k}_0$$

$$\mathfrak{Z}^{k\perp}_0 \subset \mathfrak{B}^{k\perp}_0 = \mathfrak{Z}^{*k}, \quad \mathfrak{Z}^{*k\perp}_0 \subset \mathfrak{B}^{*k\perp}_0 = \mathfrak{Z}^{k}$$

5.7.3 The Hodge decomposition

There are two Hodge decompositions, with different boundary conditions,

1. $$L^2\Lambda^k(\Omega) = \underbrace{\mathfrak{B}^k}_{\mathfrak{Z}^{*k\perp}_0} \oplus \underbrace{\mathfrak{H}^k \oplus \mathfrak{B}^{*k}_0}_{\mathfrak{Z}^{*k}_0 = \mathfrak{B}^{k\perp}} = \overbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k}^{\mathfrak{Z}^k = \mathfrak{B}^{*k\perp}_0} \oplus \overbrace{\mathfrak{B}^{*k}_0}^{\mathfrak{Z}^{k\perp}}$$

2. $$L^2\Lambda^k(\Omega) = \underbrace{\mathfrak{B}^k_0}_{\mathfrak{Z}^{*k\perp}} \oplus \underbrace{\mathfrak{H}^k_0 \oplus \mathfrak{B}^{*k}}_{\mathfrak{Z}^{*k} = \mathfrak{B}^{k\perp}_0} = \overbrace{\mathfrak{B}^k_0 \oplus \mathfrak{H}^k_0}^{\mathfrak{Z}^k_0 = \mathfrak{B}^{*k\perp}} \oplus \overbrace{\mathfrak{B}^{*k}}^{\mathfrak{Z}^{k\perp}_0}$$

5.8 Summary

$\omega^k \in \Lambda^k(\Omega)$	$k=0$	$k=1$	$k=2$	$k=3$
$\mathrm{d}\,\omega^k$	$\boldsymbol{\nabla} u$	$\nabla \times \mathbf{u}$	$\nabla \cdot \mathbf{u}$	$-$
$\delta\omega^k$	$-$	$-\nabla \cdot \mathbf{u}$	$\nabla \times \mathbf{u}$	$-\boldsymbol{\nabla} u$
$\mathfrak{i}_{\boldsymbol{\beta}}\omega^k$	$-$	$\boldsymbol{\beta} \cdot \mathbf{u}$	$\mathbf{u} \times \boldsymbol{\beta}$	$u\boldsymbol{\beta}$
$\mathfrak{j}_{\boldsymbol{\beta}}\omega^k$	$u\boldsymbol{\beta}$	$-\mathbf{u} \times \boldsymbol{\beta}$	$\boldsymbol{\beta} \cdot \mathbf{u}$	$-$
$L_{\boldsymbol{\beta}}\omega^k$	$\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u$	$\boldsymbol{\nabla}(\boldsymbol{\beta} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \boldsymbol{\beta}$	$\nabla \times (\mathbf{u} \times \boldsymbol{\beta}) + \boldsymbol{\beta}\nabla \cdot \mathbf{u}$	$\nabla \cdot (u\boldsymbol{\beta})$
$\mathcal{L}_{\boldsymbol{\beta}}\omega^k$	$-\nabla \cdot (u\boldsymbol{\beta})$	$-\nabla \times (\mathbf{u} \times \boldsymbol{\beta}) - \boldsymbol{\beta}\nabla \cdot \mathbf{u}$	$-\boldsymbol{\nabla}(\boldsymbol{\beta} \cdot \mathbf{u}) - (\nabla \times \mathbf{u}) \times \boldsymbol{\beta}$	$-\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u$
$\mathrm{tr}\,\omega^k$	$u(\mathbf{x})$	$\mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$	$\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$	$-$
$H\Lambda^k(\Omega)$	$H^1(\Omega)$	$H(\mathrm{curl}, \Omega)$	$H(\mathrm{div}, \Omega)$	$L^2(\Omega)$
V_k	$V_h(\mathrm{grad}, \Omega)$	$V_h(\mathrm{curl}, \Omega)$	$V_h(\mathrm{div}, \Omega)$	$V_h(L^2, \Omega)$

References

CHAPTER 6

DeRham sequences

here without boundary conditions

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\quad \boldsymbol{\nabla} \quad} H(\mathrm{curl}, \Omega) \xrightarrow{\quad \nabla \times \quad} H(\mathrm{div}, \Omega) \xrightarrow{\quad \nabla \cdot \quad} L^2(\Omega) \longrightarrow 0$$

6.1 Pullbacks

In the case where the physical domain $\Omega := \mathcal{F}(\hat{\Omega})$ is the *image* of a *logical* domain $\hat{\Omega}$ by a smooth mapping $\mathcal{F}$ (at least $\mathcal{C}^1$), we have the following *parallel* diagrams

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\quad \boldsymbol{\nabla} \quad} & H(\mathrm{curl}, \Omega) & \xrightarrow{\quad \nabla \times \quad} & H(\mathrm{div}, \Omega) & \xrightarrow{\quad \nabla \cdot \quad} & L^2(\Omega) \\
\imath^0 \Big\uparrow & & \imath^1 \Big\uparrow & & \imath^2 \Big\uparrow & & \imath^3 \Big\uparrow \\
H^1(\hat{\Omega}) & \xrightarrow{\quad \boldsymbol{\nabla} \quad} & H(\mathrm{curl}, \hat{\Omega}) & \xrightarrow{\quad \nabla \times \quad} & H(\mathrm{div}, \hat{\Omega}) & \xrightarrow{\quad \nabla \cdot \quad} & L^2(\hat{\Omega})
\end{array}$$

Where the *mappings* $\imath^0, \imath^1, \imath^2$ and $\imath^3$ are called **pullbacks** and are given by

$$\begin{aligned}
\phi(x) &:= \imath^0 \hat{\phi}(\hat{x}) = \hat{\phi}(\mathcal{F}^{-1}(x)) \\
\Psi(x) &:= \imath^1 \hat{\Psi}(\hat{x}) = (D\mathcal{F})^{-T} \hat{\Psi}(\mathcal{F}^{-1}(x)) \\
\Phi(x) &:= \imath^2 \hat{\Phi}(\hat{x}) = \frac{1}{J} D\mathcal{F} \hat{\Phi}(\mathcal{F}^{-1}(x)) \\
\rho(x) &:= \imath^3 \hat{\rho}(\hat{x}) = \hat{\rho}(\mathcal{F}^{-1}(x))
\end{aligned}$$

where $D\mathcal{F}$ is the **jacobian matrix** of the mapping $\mathcal{F}$.

Note: The *pullbacks* $\imath^0, \imath^1, \imath^2$ and $\imath^3$ are **isomorphisms** between the corresponding spaces.

6.2 Discrete Spaces

Let us suppose that we have a sequence of finite subspaces for each of the spaces involved in the DeRham sequence. The discrete DeRham sequence stands for the following commutative diagram between continuous and discrete spaces

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{grad}}"'] & H(\mathrm{curl}, \Omega) \arrow[r, "\nabla\times"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{curl}}"'] & H(\mathrm{div}, \Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{div}}"'] & L^2(\Omega) \arrow[d, "\boldsymbol{\Pi}_h^{L^2}"'] \\
V_h(\mathrm{grad}, \Omega) \arrow[r, "\boldsymbol{\nabla}"] & V_h(\mathrm{curl}, \Omega) \arrow[r, "\nabla\times"] & V_h(\mathrm{div}, \Omega) \arrow[r, "\nabla\cdot"] & V_h(L^2, \Omega)
\end{tikzcd}$$

When using a Finite Elements methods, we often deal with a reference element, and thus we need also to apply the *pullbacks* on the discrete spaces. In fact, we have again the following parallel diagram

$$\begin{tikzcd}
V_h(\mathrm{grad}, \Omega) \arrow[r, "\boldsymbol{\nabla}"] & V_h(\mathrm{curl}, \Omega) \arrow[r, "\nabla\times"] & V_h(\mathrm{div}, \Omega) \arrow[r, "\nabla\cdot"] & V_h(L^2, \Omega) \\
V_h(\mathrm{grad}, \hat{\Omega}) \arrow[r, "\boldsymbol{\nabla}"] \arrow[u, "\imath^0"] & V_h(\mathrm{curl}, \hat{\Omega}) \arrow[r, "\nabla\times"] \arrow[u, "\imath^1"] & V_h(\mathrm{div}, \hat{\Omega}) \arrow[r, "\nabla\cdot"] \arrow[u, "\imath^2"] & V_h(L^2, \hat{\Omega}) \arrow[u, "\imath^3"]
\end{tikzcd}$$

Since, the *pullbacks* are **isomorphisms** in the previous diagram, we can define a **one-to-one** correspondance

$$\begin{aligned}
\phi &:= \imath^0 \hat{\phi}, \quad \phi \in V_h(\mathrm{grad}, \Omega), \hat{\phi} \in V_h(\mathrm{grad}, \hat{\Omega}) \\
\Psi &:= \imath^1 \hat{\Psi}, \quad \Psi \in V_h(\mathrm{curl}, \Omega), \hat{\Psi} \in V_h(\mathrm{curl}, \hat{\Omega}) \\
\Phi &:= \imath^2 \hat{\Phi}, \quad \Phi \in V_h(\mathrm{div}, \Omega), \hat{\Phi} \in V_h(\mathrm{div}, \hat{\Omega}) \\
\rho &:= \imath^3 \hat{\rho}, \quad \rho \in V_h(L^2, \Omega), \hat{\rho} \in V_h(L^2, \hat{\Omega})
\end{aligned}$$

We have then, the following results

$$\begin{aligned}
\boldsymbol{\nabla} \phi &= \imath^1 \boldsymbol{\nabla} \hat{\phi}, \quad \phi \in V_h(\mathrm{grad}, \Omega) \\
\nabla \times \Psi &= \imath^2 \nabla \times \hat{\Psi}, \quad \Psi \in V_h(\mathrm{curl}, \Omega) \\
\nabla \cdot \Phi &= \imath^3 \nabla \cdot \hat{\Phi}, \quad \Phi \in V_h(\mathrm{div}, \Omega)
\end{aligned}$$

6.3 Projectors

In some cases, one may need to define projectors on *smooth* functions

$$\begin{tikzcd}
\mathcal{C}^\infty(\Omega) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{grad}}"'] & \mathcal{C}^\infty(\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{curl}}"'] & \mathcal{C}^\infty(\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\boldsymbol{\Pi}_h^{\mathrm{div}}"'] & \mathcal{C}^\infty(\Omega) \arrow[d, "\boldsymbol{\Pi}_h^{L^2}"'] \\
V_h(\mathrm{grad}, \Omega) \arrow[r, "\boldsymbol{\nabla}"] & V_h(\mathrm{curl}, \Omega) \arrow[r, "\nabla\times"] & V_h(\mathrm{div}, \Omega) \arrow[r, "\nabla\cdot"] & V_h(L^2, \Omega)
\end{tikzcd}$$

6.4 Discrete DeRham sequence for B-Splines

Buffa et al [BSV09] show the construction of a discrete DeRham sequence using B-Splines, (here without boundary conditions)

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\nabla} & H(\text{curl},\Omega) & \xrightarrow{\nabla\times} & H(\text{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
\Big\downarrow \boldsymbol{\Pi}_h^{\text{grad}} & & \Big\downarrow \boldsymbol{\Pi}_h^{\text{curl}} & & \Big\downarrow \boldsymbol{\Pi}_h^{\text{div}} & & \Big\downarrow \boldsymbol{\Pi}_h^{L^2} \\
\mathcal{S}^{p,p,p} & \xrightarrow{\nabla} & \begin{pmatrix} \mathcal{S}^{p-1,p,p} \\ \mathcal{S}^{p,p-1,p} \\ \mathcal{S}^{p,p,p-1} \end{pmatrix} & \xrightarrow{\nabla\times} & \begin{pmatrix} \mathcal{S}^{p,p-1,p-1} \\ \mathcal{S}^{p-1,p,p-1} \\ \mathcal{S}^{p-1,p-1,p} \end{pmatrix} & \xrightarrow{\nabla\cdot} & \mathcal{S}^{p-1,p-1,p-1}
\end{array}$$

6.4.1 1d case

1. DeRham sequence is reduced to

$$\mathbb{R} \hookrightarrow \underbrace{\mathcal{S}^p}_{V_h(\text{grad},\hat{\Omega})} \xrightarrow{\nabla} \underbrace{\mathcal{S}^{p-1}}_{V_h(L^2,\hat{\Omega})} \longrightarrow 0$$

2. The recursion formula for derivative writes

$$N_i^{p\,\prime}(t) = D_i^p(t) - D_{i+1}^p(t) \quad \text{where} \quad D_i^p(t) = \frac{p}{t_{i+p+1} - t_i} N_i^{p-1}(t)$$

3. we have $\mathcal{S}^{p-1} = \mathbf{span}\{N_i^{p-1}, 1 \leq i \leq n-1\} = \mathbf{span}\{D_i^p, 1 \leq i \leq n-1\}$ which is a change of basis as a diagonal matrix
4. Now if $u \in S^p$, with and expansion $u = \sum_i u_i N_i^p$, we have

$$u' = \sum_i u_i\,(N_i^p)' = \sum_i (-u_{i-1} + u_i) D_i^p$$

5. If we introduce the B-Splines coefficients vector $\mathbf{u} := (u_i)_{1 \leq i \leq n}$ (and $\mathbf{u}^\star$ for the derivative), we have

$$\mathbf{u}^\star = D\mathbf{u}$$

where D is the incidence matrix (of entries -1 and $+1$)

Discrete derivatives:

$$\mathcal{G} = D$$

6.4.2 2d case

In *2d*, the are two De-Rham complexes:

$$\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\nabla} & H(\text{curl},\Omega) & \xrightarrow{\nabla\times} & L^2(\Omega) \\
\Big\downarrow \boldsymbol{\Pi}_h^{\text{grad}} & & \Big\downarrow \boldsymbol{\Pi}_h^{\text{curl}} & & \Big\downarrow \boldsymbol{\Pi}_h^{L^2} \\
V_h(\text{grad},\Omega) & \xrightarrow{\nabla} & V_h(\text{curl},\Omega) & \xrightarrow{\nabla\times} & V_h(L^2,\Omega)
\end{array}$$

and

$$
\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\nabla\times} & H(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
\Big\downarrow \mathbf{\Pi}_h^{\mathrm{grad}} & & \Big\downarrow \mathbf{\Pi}_h^{\mathrm{div}} & & \Big\downarrow \mathbf{\Pi}_h^{L^2} \\
V_h(\mathrm{grad},\Omega) & \xrightarrow{\boldsymbol{\nabla}} & V_h(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & V_h(L^2,\Omega)
\end{array}
$$

Let I be the identity matrix, we have

Discrete derivatives:

$$
\mathcal{G} = \begin{pmatrix} D\otimes I \\ I\otimes D \end{pmatrix}
$$

$$
\mathcal{C} = \begin{pmatrix} I\otimes D \\ -D\otimes I \end{pmatrix} \quad \text{[scalar curl]}, \quad \mathcal{C} = \begin{pmatrix} -I\otimes D & D\otimes I \end{pmatrix} \quad \text{[vectorial curl]}
$$

$$
\mathcal{D} = \begin{pmatrix} D\otimes I & I\otimes D \end{pmatrix}
$$

6.4.3 3d case

Discrete derivatives:

$$
\mathcal{G} = \begin{pmatrix} D\otimes I\otimes I \\ I\otimes D\otimes I \\ I\otimes I\otimes D \end{pmatrix}
$$

$$
\mathcal{C} = \begin{pmatrix} 0 & -I\otimes I\otimes D & I\otimes D\otimes I \\ I\otimes I\otimes D & 0 & -D\otimes I\otimes I \\ -I\otimes D\otimes I & D\otimes I\otimes I & 0 \end{pmatrix}
$$

$$
\mathcal{D} = \begin{pmatrix} D\otimes I\otimes I & I\otimes D\otimes I & I\otimes I\otimes D \end{pmatrix}
$$

Note: From now on, we will denote the discrete derivative by $\mathbb{D}_k$ for the one going from V_k to V_{k+1}.

6.5 Algebraic identities

Let us consider the discretization of the exterior derivative

$$
\omega^{k+1} = \mathrm{d}\,\omega^k
$$

multiplying by a test function η^{k+1} and integrating over the whole computation domain, we get

$$
\left(\eta^{k+1}, \omega^{k+1}\right)_{k+1} = \left(\eta^{k+1}, \mathrm{d}\,\omega^k\right)_{k+1}
$$

let E^{k+1}, W^k and W^{k+1} be the vector representation of η^{k+1}, ω^k and ω^{k+1}. We get

$$E^{k+1^T} M_{k+1} W^{k+1} = E^{k+1^T} D_{k+1,k} W^k$$

where

$$D_{k+1,k} = \left(\left(\eta_i^{k+1}, \mathrm{d}\,\omega_j^k\right)_{k+1} \right)_{i,j}$$

On the other hand, using the coderivative, we get

$$\left(\eta^{k+1}, \omega^{k+1}\right)_{k+1} = \left(\delta\eta^{k+1}, \omega^k\right)_k + BC$$

Let us now introduce the following matrix

$$D_{k,k+1} = \left(\left(\delta\eta_i^{k+1}, \omega_j^k\right)_k \right)_{i,j}$$

hence,

$$E^{k+1^T} D_{k,k+1} W^k = \left(\mathbb{D}_{k+1}^\star E^{k+1}\right)^T M_k W^k$$

Therefor, we have the following important result

Proposition:

- $D_{k+1,k} = D_{k,k+1} + BC$
- $D_{k+1,k} = M_{k+1} \mathbb{D}_k^T$
- $D_{k,k+1} = {\mathbb{D}_{k+1}^\star}^T M_k$

References

CHAPTER 7

API

You will find here both the *Fortran* **doxygen** documentation as well as the *Python-API*.

7.1 Fortran API

7.2 Python API

7.2.1 spl package

Subpackages

spl.core package

Submodules

spl.core.basic module

Module contents

Submodules

spl.mapping module

spl.utilities module

Module contents

CHAPTER 8

Indices and tables

- genindex
- modindex
- search

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